# J.D. Jackson Problem 9.12

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## 1 Finding the multipole moments

We have to start by finding the charge density by setting its integral equal to the total charge.

$$Q = \iiint \rho r^2 \sin\theta dr d\theta d\phi \tag{1}$$

$$\frac{Q}{2\pi} = \rho \int_0^\pi \int_0^{R(\theta)} r^2 \sin\theta dr d\theta d\phi \tag{2}$$

$$\frac{Q}{2\pi} = \frac{\rho}{3} \int_0^\pi r^3\big|_0^{R(\theta)} \sin\theta d\theta \tag{3}$$

$$\frac{3Q}{2\pi} = \rho R_0^3 \int_0^\pi (1+\beta P_2(\cos\theta))^3 \sin\theta d\theta \tag{4}$$

$$\frac{3Q}{2\pi} = \rho R_0^3 \int_{-1}^1 (1+\beta P_2 x)^3 dx \tag{5}$$

$$\frac{12Q}{2\pi} = \rho R_0^3 \int_{-1}^1 (2+3\beta x^2 - \beta)^3 dx \tag{6}$$

The integrand expands and integrates as follows

$$\rho = \frac{6Q}{\pi R_0^3}\left[(2-\beta)^3 + \frac{1}{3}(9\beta(2-\beta)^2) + \frac{1}{5}(27\beta^2(2-\beta)) + \frac{1}{7}27\beta^3\right]^{-1} \tag{7}$$

Keeping only the lower order terms in beta,

$$\rho = \frac{15Q}{4\pi R_0^3(5+3\beta^2)} \tag{8}$$

Now that we have the charge density we can find the multipole moments.

$$Q_{lm} = \int r^l Y_{lm}^* \rho d^3x \tag{9}$$

$$Q_{l0} = \frac{15Q}{4\pi R_0^3(5+3\beta^2)}\sqrt{\frac{2l+1}{4\pi}}\frac{2\pi}{3+l}\int_0^\pi r^{3+l}\big|_0^{R(\theta)} \sin\theta P_l(\cos\theta) d\theta \tag{10}$$

$$Q_{l0} = \frac{15Q}{4\pi R_0^3(5+3\beta^2)}\sqrt{\frac{2l+1}{4\pi}}\frac{2\pi}{3+l}R_0^{3+l}\int_{-1}^1 (1+\beta P_2(x))^{3+l} P_l(x) dx \tag{11}$$

So, let's calculate the first few of these.

$$Q_{00} = \frac{15Q}{4\pi R_0^3(5+3\beta^2)}\sqrt{\frac{2\cdot 1+1}{4\pi}}\frac{2\pi}{3+0}R_0^3\frac{2}{5}(3\beta^2+5) \tag{12}$$

$$Q_{00} = \frac{Q}{\sqrt{4\pi}} \tag{13}$$

$$Q_{10} = 0 \tag{14}$$

$$Q_{20} = \frac{15Q}{4\pi R_0^3(5+3\beta^2)}\sqrt{\frac{2\cdot 2+1}{4\pi}}\frac{2\pi}{3+2}R_0^5\frac{2\beta}{1001}(572\beta^2+1001) \tag{15}$$

$$Q_{20} = \sqrt{\frac{9}{20\pi}}QR_0^2\beta \tag{16}$$

## 2 Power per solid angle

We can find the power radiated by each multipole using equation 9.151.

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{Z_0}{2k^2}\left|a(l,m)\right|^2\left|\mathbf{X}_{lm}\right|^2 \tag{17}$$

According to the footnote on page 431, $\mathbf{X}_{00} = 0$, so we see that the monopole moment doesn't radiate. The $l = 1$ term also does not radiate because its $a(1,0)$ term is zero. The other $a$ terms will need to be calculated more rigorously. We'll use the long wavelength limit as described in equation 9.169. (Our $Q'_{lm}$'s are zero because there is no magnetization.

$$a(l,m) \approx \frac{ck^{l+2}}{i(2l+1)!!}\sqrt{\frac{l+1}{l}}Q_{lm} \tag{18}$$

$$a(2,0) \approx \frac{ck^4}{15i}\sqrt{3/2}\sqrt{\frac{9}{20\pi}}QR_0^2\beta \tag{19}$$

$$\left|a(2,0)\right|^2 = \frac{3}{1000\pi}c^2k^8Q^2R_0^4\beta^2 \tag{20}$$

The table on page 437 give us the last remaining piece of the puzzle

$$\left|\mathbf{X}_{20}\right|^2 = \frac{15}{8\pi}\sin^2\theta\cos^2\theta \tag{21}$$

Putting all of this together, we see,

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{9Z_0c^2k^6Q^2R_0^4\beta^2}{3200\pi^2}\sin^2\theta\cos^2\theta \tag{22}$$

## 3 Total Power

Total power is given by integrating the power per solid angle over all possible solid angles.

$$P = \frac{9Z_0c^2k^6Q^2R_0^4\beta^2}{1600\pi}\int_0^\pi \sin^3\theta\cos^2\theta\,\mathrm{d}\theta \tag{23}$$

$$P = \frac{3Z_0c^2k^6Q^2R_0^4\beta^2}{2000\pi} \tag{24}$$