I’ll begin this problem by assuming that $\Phi$ is of a separable form, $\Phi(r, \theta, z) = R(r)\Theta(\theta)Z(z)$ in cylindrical coordinates. Applying the Laplacian operator gives three ordinary differential equations subject to five constraints.

Five boundary constraints for separated solutions:

1. $\Phi = 0 \Rightarrow \Phi(z) = 0$ (1a)
2. $\Phi = 0 \Rightarrow \Phi(z) = L$ (1b)
3. $\Phi = \pm \infty \Rightarrow \Theta(\theta) = 0$ (1c)
4. $\Phi = V(\theta, z) \Rightarrow \Phi(r, 0, z) = \Phi(r, 2\pi, z)$ (1d)
5. $\Phi(r, 0, z) = \Phi(r, 2\pi, z)$ (1e)

Solving each of the three ordinary differential equations and naming the constants in the style of Jackson section 3.7 we have:

2. $\frac{Z''(z)}{Z(z)} = -k^2 \Rightarrow Z(z) = A_z \sin(kz) + B_z \cos(kz)$

Constraints 1 and 2 show that $B_z = 0$ and $k = \frac{n\pi}{L}$ where $n$ is a positive integer.

3. $\frac{\Theta''(\theta)}{\Theta(\theta)} = -\nu^2 \Rightarrow \Theta(\theta) = A_\theta \sin(\nu\theta) + B_\theta \cos(\nu\theta)$

Constraint 5 shows that $\nu$ is a non-negative integer.

4. $R'' + \frac{R'}{r} - \left( k^2 + \frac{\nu^2}{r^2} \right) R = 0$

Here the primes denote derivative with respect to $r$. Now, we’ll follow Jackson and make the substitution $x = kr$, $dx = kdr$

$\frac{\dot{R} + \frac{\dot{R}}{x} - \left( 1 + \frac{\nu^2}{x^2} \right) R = 0 \Rightarrow R(x) = A_r I_\nu(x) + B_r K_\nu(x)$ (5)

Constraint 3 shows that $B_r = 0$.

So putting the whole solution back together:

$$\Phi = \sum_{n,\nu} I_\nu \left( \frac{n\pi r}{L} \right) \sin \left( \frac{n\pi z}{L} \right) \left[ A_{n\nu} \sin(\nu\theta) + B_{n\nu} \cos(\nu\theta) \right]$$ (6)

All that is left to do now is calculate the $A_{n\nu}$’s and $B_{n\nu}$’s which can be accomplished by applying the final boundary condition.

$$V(\theta, z) = \sum_{n,\nu} I_\nu \left( \frac{n\pi b}{L} \right) \sin \left( \frac{n\pi z}{L} \right) \left[ A_{n\nu} \sin(\nu\theta) + B_{n\nu} \cos(\nu\theta) \right]$$ (7)

Multiply both sides by $\cos(\nu'\theta)$ and integrate

$$\int_0^{2\pi} V(\theta, z) \cos(\nu'\theta)d\theta = \sum_{n,\nu} I_\nu \left( \frac{n\pi b}{L} \right) \sin \left( \frac{n\pi z}{L} \right) \int_0^{2\pi} A_{n\nu} \sin(\nu\theta) \cos(\nu'\theta) + B_{n\nu} \cos(\nu\theta) \cos(\nu'\theta)d\theta$$ (8)
The $A_{n\nu}$ terms are all zero because the sine and cosine product integrates to zero for any values of $\nu$ and $\nu'$. The $B_{n\nu}$ terms integrate to zero in every case except when $\nu = \nu'$.

\[
\int_0^{2\pi} V(\theta, z) \cos(\nu' \theta) d\theta = \sum_{n, \nu} I_{\nu} \left( \frac{n\pi b}{L} \right) \sin \left( \frac{n\pi z}{L} \right) \pi B_{n\nu} \delta_{\nu\nu'}
\]  

(9)

The kronecker delta collapses the sum over $\nu$

\[
\int_0^{2\pi} V(\theta, z) \cos(\nu' \theta) d\theta = \pi \sum_n B_{n\nu'} I_{\nu'} \left( \frac{n\pi b}{L} \right) \sin \left( \frac{n\pi z}{L} \right)
\]  

(10)

Multiply both sides by $\sin\left(\frac{n'\pi z}{L}\right)$ and integrate

\[
\int_0^L \int_0^{2\pi} V(\theta, z) \cos(\nu' \theta) \sin \left( \frac{n'\pi z}{L} \right) d\theta dz = \pi \sum_n B_{n\nu'} I_{\nu'} \left( \frac{n\pi b}{L} \right) \int_0^L \sin \left( \frac{n\pi z}{L} \right) \sin \left( \frac{n'\pi z}{L} \right) dz
\]  

(11)

As with the cosines above, the sines are orthogonal and the integral is only non-zero when $n = n'$.

\[
\int_0^L \int_0^{2\pi} V(\theta, z) \cos(\nu' \theta) \sin \left( \frac{n'\pi z}{L} \right) d\theta dz = \pi \sum_n B_{n\nu'} I_{\nu'} \left( \frac{n\pi b}{L} \right) L \frac{L}{2} \delta_{nn'}
\]  

(12)

The kronecker delta collapses the sum over $n$

\[
\int_0^L \int_0^{2\pi} V(\theta, z) \cos(\nu' \theta) \sin \left( \frac{n'\pi z}{L} \right) d\theta dz = \frac{\pi L}{2} B_{n\nu'} I_{\nu'} \left( \frac{n'\pi b}{L} \right)
\]  

(13)

Finally, solving for $B_{n\nu}$ and noting that the $A_{n\nu}$'s can be calculated in the exact same way, we have,

\[
A_{n\nu} = \frac{2}{L\pi I_{\nu}} \left( \frac{n\pi b}{L} \right) \int_0^{2\pi} \int_0^L \sin \left( \frac{n\pi z}{L} \right) \sin(\nu \theta) V(\theta, z) dz d\theta
\]  

(14a)

\[
B_{n\nu} = \frac{2}{L\pi I_{\nu}} \left( \frac{n\pi b}{L} \right) \int_0^{2\pi} \int_0^L \sin \left( \frac{n\pi z}{L} \right) \cos(\nu \theta) V(\theta, z) dz d\theta
\]  

(14b)