# J.D. Jackson Problem 3.6

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## 1 Expansion in Spherical Harmonics

Begin with the known form of electric potential for point charges.

$$\Phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{a}|} - \frac{1}{|\mathbf{r}+\mathbf{a}|}\right) \tag{1}$$

Expand into spherical harmonics using equation 3.70.

$$\Phi = \frac{q}{\epsilon_0}\sum_{l,m}\left[\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_lm^*(0,\phi')Y_lm(\theta,\phi) - \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_lm^*(\pi,\phi')Y_lm(\theta,\phi)\right] \tag{2}$$

$$\Phi = \frac{q}{\epsilon_0}\sum_{l,m}\frac{r_<^l Y_{lm}(\theta,\phi)}{r_>^{l+1}(2l+1)}\left[Y_{lm}^*(0,\phi') - Y_{lm}^*(\pi,\phi')\right] \tag{3}$$

We know that m=0 by azimutal symmetry of the source distribution

$$\Phi = \frac{q}{\epsilon_0}\sum_{l,m}\frac{r_<^l Y_{l0}^*(\theta,\phi)}{r_>^{l+1}(2l+1)}\left[Y_{l0}^*(0,\phi') - Y_{l0}^*(\pi,\phi')\right] \tag{4}$$

Useful identities for these spherical harmonics:

$$Y_{l0}^*(0,\phi') = Y_{l0}(0,\phi') = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos 0) = \sqrt{\frac{2l+1}{4\pi}} \tag{5a}$$

$$Y_{l0}^*(\pi,\phi') = Y_{l0}(\pi,\phi') = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos \pi) = \sqrt{\frac{2l+1}{4\pi}}(-1)^l \tag{5b}$$

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta) \tag{5c}$$

Substituting the above identities,

$$\Phi = \frac{q}{\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}(2l+1)}\sqrt{\frac{2l+1}{4\pi}}^2 P_l(\cos\theta)\left[1-(-1)^l\right] \tag{6}$$

$$\boxed{\Phi = \frac{q}{2\pi\epsilon_0}\sum_{l-odd}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)} \tag{7}$$

## 2 Limit of a Dipole

Simply taking the limit as $a \to 0$ will represent putting two point charges directly on top of each other. We already know the potential of that configuration. Instead, we will keep the product $[qa]$ constant. In this limit, it only makes sense to treat the case $[r > a]$

$$\Phi = \frac{q}{2\pi\epsilon_0}\sum_{lodd}\frac{a^l}{r^{l+1}}P_l(\cos\theta) \tag{8}$$

We want to keep the product $qa$ constant, so we have to factor an a outside the sum.

$$\Phi = \frac{qa}{2\pi\epsilon_0}\sum_{lodd}\frac{a^{l-1}}{r^{l+1}}P_l(\cos\theta) \tag{9}$$

Put in the dipole definition

$$\Phi = \frac{p}{4\pi\epsilon_0}\sum_{lodd}\frac{a^{l-1}}{r^{l+1}}P_l(\cos\theta) \tag{10}$$

In the limit as $a \to 0$, all term in the sum also approach zero except the $l = 0$ term.

$$\lim_{a\to 0}\Phi = \frac{p}{4\pi\epsilon_0 r^2}P_1(\cos\theta) \tag{11}$$

$$\boxed{\lim_{a\to 0}\Phi = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}} \tag{12}$$

## 3 Enclosing the dipole in a grounded sphere

We are now imposing the condition that the potential must be zero on the surface of a sphere. We know the general form of the solution for potential inside a sphere with a known potential, so we will use it to perfectly cancel the potential that would have otherwise been induced by the dipole.

$$\Phi_h = \sum_l [D_l r^l + B_l R^{-l-1}P_l(\cos\theta) \tag{13}$$

Because we can't have the potential blowing up at the origin,

$$\Phi_h = \sum_l D_l r^l P_l(\cos\theta) \tag{14}$$

We are going to add these two solutions so that the boundary condition on the sphere is met,

$$\Phi_{total} = \Phi_h + \Phi_{dip} = \sum_l D_l r^l P_l(\cos\theta) + \frac{p}{4\pi\epsilon_0}P_1(\cos(\theta)) \tag{15}$$

To make the algebra a little easier, I'll redefine the$D_l$'s as $A_l$'s.

$$\Phi_{total} = \frac{p}{4\pi\epsilon_0}\left[\sum_l -A_l r^l P_l(\cos\theta) + \frac{1}{r^2}P_1(\cos\theta)\right] \tag{16}$$

Now I'll apply the boundary condition at the surface of the sphere.

$$\Phi_{total}(b,\theta) = 0 = \sum_l -A_l b^l P_l(\cos\theta) + \frac{1}{b^2}P_1(\cos\theta) \tag{17}$$

$$\sum_l A_l b^{l+2} P_l(\cos\theta) = P_1(\cos\theta) \tag{18}$$

We know that the $P_l$'s are orthogonal so,

$$A_1 b^3 P_1(\cos\theta) = P_1(\cos\theta) \tag{19}$$

So we've now found all the of $A_l$'s of which only one is non zero.

$$A_1 = \frac{1}{b^3} \tag{20}$$

Putting it all back together,

$$\boxed{\Phi_{total} = \frac{p\cos\theta}{4\pi\epsilon_0}\left[\frac{1}{r^2} - \frac{r}{b^3}\right]} \tag{21}$$