Goldstein, Poole, and Safko Problem 1.19

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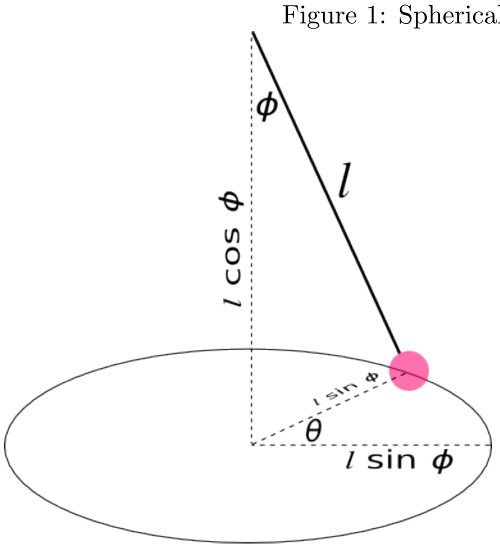
Figure 1: Spherical pendulum with coordinates labelled



Establishing the coordinate system as it is labelled in the figure, we can begin by writing the potential and kinetic energy functions.

$$U = -mgl\cos\phi \tag{1}$$

$$T = \frac{1}{2}m\left(l^2\dot{\theta}^2\sin^2\phi + l^2\dot{\phi}^2\right) \tag{2}$$

$$T = \frac{ml^2}{2}\left(\dot{\theta}^2\sin^2\phi + \dot{\phi}^2\right) \tag{3}$$

That being established,we can write the Lagrangian as $L = T - U$.

$$L = \frac{ml^2}{2}\left(\dot{\theta}^2\sin^2\phi + \dot{\phi}^2\right) + mgl\cos\phi \tag{4}$$

Now that we have the Lagrangian in all of its glory, we can compute the appropriate derivatives that we will need to find the equations of motion.

$$\begin{array}{ll}
\frac{\partial L}{\partial \theta} = 0 & \frac{\partial L}{\partial \phi} = ml^2\dot{\theta}^2\sin\phi\cos\phi \\
\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}\sin^2\phi & \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ml^2\left(\ddot{\theta}\sin^2\phi + 2\dot{\theta}\dot{\phi}\sin\phi\right) & \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = ml^2\ddot{\phi}
\end{array}$$

Now we can use the Euler Lagrange equation to determine the equations of motion. The ϕ equation becomes:

$$ml^2\dot{\theta}^2\sin\phi\cos\phi - mgl\sin\phi = ml^2\ddot{\phi} \tag{5}$$

$$\ddot{\phi} - \dot{\theta}^2 \sin\phi \cos\phi + \frac{g}{l} \sin\phi = 0 \tag{6}$$

The Euler Lagrange equation for $/theta$ is:

$$ml^2 \left(\ddot{\theta} \sin^2 \phi + 2\dot{\theta}\dot{\phi} \sin\phi \right) = 0 \tag{7}$$

What this really tells us is that the quantity $\frac{\partial L}{\partial \dot{\theta}}$ is conserved.

These equations are not solvable analytically, but could be solved numerically.